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Phase Operator for the Photon Field and an Index Theorem

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Abstract

An index relation $dim\ ker\ a^\dagger a - dim\ ker\ aa^\dagger = 1$ is satisfied by the creation and annihilation operators $a^\dagger$ and a of a harmonic oscillator. A hermitian phase operator, which inevitably leads to $dim\ ker\ a^\dagger a - dim\ ker\ aa^\dagger = 0$, cannot be consistently defined. If one considers an $s+1$ dimensional truncated theory, a hermitian phase operator of Pegg and Barnett which carries a vanishing index can be defined. However, for arbitrarily large s, we show that the vanishing index of the hermitian phase operator of Pegg and Barnett causes a substantial deviation from minimum uncertainty in a characteristically quantum domain with small average photon numbers. We also mention an interesting analogy between the present problem and the chiral anomaly in gauge theory which is related to the Atiyah-Singer index theorem. It is suggested that the phase operator problem related to the above analytic index may be regarded as a new class of quantum anomaly. From an anomaly view point ,it is not surprising that the phase operator of Susskind and Glogower, which carries a unit index, leads to an anomalous identity and an anomalous commutator.

1 Introduction

The quantum phase operator has been studied by various authors in the past[1$\sim$ 10]. We here add yet another remark on this much studied subject, in particular, the absence of a hermitian phase operator and the lack of a mathematical basis for $\Delta N \Delta \phi \geq 1/2$, on the basis of a notion of index or an index theorem. To be specific, we study the simplest one-dimensional harmonic oscillator defined by

$$\begin{aligned} h &= \frac{1}{2}(a^{\dagger}a + aa^{\dagger}) \\ &= a^{\dagger}a + 1/2 \end{aligned} \tag{1}$$

where a and $a^{\dagger}$ stand for the annihilation and creation operators satisfying the standard commutator

$$[a, a^{\dagger}] = 1 \tag{2}$$

The vacuum state $|0\rangle$ is annihilated by a

$$a|0\rangle = 0 \tag{3}$$

which ensures the absence of states with negative norm. The number operator defined by

$$N = a^{\dagger}a \tag{4}$$

then has non-negative integers as eigenvalues, and the annihilation operator a is represented by

$$a = |0\rangle\langle 1| + |1\rangle\langle 2|\sqrt{2} + |2\rangle\langle 3|\sqrt{3} + ... \tag{5}$$

in terms of the eigenstates $|k\rangle$ of the number operator

$$N|k\rangle = k|k\rangle \tag{6}$$

with $k = 0, 1, 2, ...$ The creation operator $a^\dagger$ is given by the hermitian conjugate of a in (5).

The notion of index or an index theorem provides a powerful machinery for an analysis of the representation of linear operators such as a and $a^\dagger$. In the representation of a and $a^\dagger$ specified above we have the index condition[11]

$$dim\ ker\ a^\dagger a - dim\ ker\ aa^\dagger = 1 \tag{7}$$

where $dim\ ker\ a^\dagger a$,for example, stands for the number of normalizable basis vectors u_n which satisfy $a^\dagger a u_n = 0$;$dim\ ker\ a^\dagger a$ thus agrees with the number of zero eigenvalues of the hermitian operator $a^\dagger a$. To assign a sensible meaning to the index relation, we need a well-defined kernel with

$$dim\ ker\ a^\dagger a < \infty \tag{8}$$

since $dim\ ker\ a^\dagger a = \infty$ corresponds to a singular point of the index relation, which takes place in a quantum deformation of the oscillator algebra [12], for example. To be mathematically precise, it is also important to confirm that the operator $a^\dagger a$ has discrete eigenvalues. In the conventional notation of index theory, the relation (7) is written by using the trace of well-defined operators as

$$Tr(e^{-a^\dagger a/M^2}) - Tr(e^{-aa^\dagger/M^2}) = 1 \tag{9}$$

with M^2 standing for a positive constant. If $Tr(e^{-a^\dagger a/M^2})$ diverges,one needs a different regularization scheme.

The relation(9) is confirmed for the standard representation(5) as

$$1 + (\sum_{n=1}^{\infty} e^{-n/M^2}) - (\sum_{n=1}^{\infty} e^{-n/M^2}) = 1 \tag{10}$$

independently of the value of M^2.

If one should suppose the existence of a well defined hermitian phase operator ϕ, one would have a polar decomposition

$$a = U(\phi)H = e^{i\phi}H \tag{11}$$

as was originally suggested by Dirac[1]. Here U and H stand for unitary and hermitian operators, respectively. To avoid the complications related to the periodicity problem, we exclusively deal with the operator of the form $e^{i\phi}$ in this paper; a hermitian phase operator thus means a unitary $e^{i\phi}$, and a non-hermitian phase operator means a non-unitary $e^{i\phi}$.

If (11) should be valid, one has

$$aa^{\dagger} = UH^2U^{\dagger} \tag{12}$$

which is unitary equivalent to $a^{\dagger}a = H^2$; $a^{\dagger}a$ and $aa^{\dagger}$ thus have an identical number of zero eigenvalues. In this case, we have in the same notation as (9)

$$\begin{aligned} & Tr(e^{-a^{\dagger}a/M^2}) - Tr(e^{-aa^{\dagger}/M^2}) \\ = \; & Tr(e^{-H^2/M^2}) - Tr(e^{-UH^2U^{\dagger}/M^2}) = 0 \end{aligned} \tag{13}$$

This relation when combined with (9) constitutes a proof of the absence of a hermitian phase operator in the framework of index theory.

The basic utility of the notion of index or an index theorem lies in the fact that the index as such is an integer and remains invariant under a wide class of continuous deformation. If one denotes two unitary equivalent operators by a and a', one has the relation

$$a = V^{\dagger}a'U \tag{14}$$

with V and U two unitary matrices. For the operator with a non-zero index , the left and right vector spaces may in general be different, which is the origin of the appearance of two unitary matrices V and U in (14). If (14) is valid, one has the relations

$$\begin{aligned} a^\dagger a &= U^\dagger a'^\dagger a' U \\ a a^\dagger &= V^\dagger a' a'^\dagger V \end{aligned} \tag{15}$$

and thus $dim\ ker\ a^\dagger a = dim\ ker\ a'^\dagger a'$ and $dim\ ker\ aa^\dagger = dim\ ker\ a'a'^\dagger$,namely, a and a' have an identical index. As an example, the unitary time development of a and $a^\dagger$ dictated by the Heisenberg equation of motion, which includes a fundamental phenomenon such as squeezing, does not alter the index relation. Another consequence is that one cannot generally relate the representation spaces of annihilation operators with <u>different</u> indices by a unitary transformation.

If one truncates the representation of a to any finite dimension, for example, to an $(s+1)$ dimension with s a positive integer, one obtains

$$dim\ ker\ a_s^\dagger a_s - dim\ ker\ a_s a_s^\dagger \ = \ 0 \tag{16}$$

where a_s stands for an $s+1$ dimensional truncation of a. This relation (16) is proved by noting that non-vanishing eigenvalues of $a_s^\dagger a_s$ and $a_s a_s^\dagger$ are in one-to-one correspondence: In the eigenvalue equations

$$a_s^\dagger a_s u_n = \lambda_n^2 u_n \tag{17}$$

one may define $v_n = a_s u_n / \lambda_n$ for $\lambda_n \neq 0$. One then obtains

$$a_s a_s^\dagger v_n = \lambda_n^2 v_n \tag{18}$$

by multiplying a_s to both hand sides of eq.(17). The eigen functions v_n are correctly normalized if u_n are normalized. For any finite dimensional matrix , one also has

$$Tr(a_s^\dagger a_s) = Tr(a_s a_s^\dagger) \tag{19}$$

These two facts combined then lead to the statement (16), namely, $a_s^\dagger a_s$ and $a_s a_s^\dagger$ contain the same number of zero eigenvalues.

From the above analysis of the index condition , the polar decomposition (11) could be consistently defined if one truncates the dimension of the representation space to a finite $s+1$ dimension . This is the approach adopted by Pegg and Barnett[8] in their definition of a hermitian phase operator. See also Ref.[9]. Their basic idea is then to let s arbitrarily large later. If one denotes the truncated operators by a_s and $a_s^\dagger$, one obtains the relation (16), namely

$$\begin{aligned} & dim\ ker\ a_s^\dagger a_s - dim\ ker\ a_s a_s^\dagger \\ = \ & Tr_{(s+1)}(e^{-a_s^\dagger a_s/M^2}) - Tr_{(s+1)}(e^{-a_s a_s^\dagger/M^2}) = 0 \end{aligned} \tag{20}$$

where $Tr_{(s+1)}$ denotes an $s+1$ dimensional trace. This index relation holds independently of M^2 and s.

One technical but important point to be noted here is that the trace in (20) is defined in an $s+1$ dimensional space, whereas the trace in (9) is defined in an infinite dimensional space. To directly compare these two relations, one may extend the operator a_s to a suitable infinite dimensional A_s in such a way that

$$Tr(e^{-A_s^\dagger A_s/M^2}) - Tr(e^{-A_s A_s^\dagger/M^2}) = 0 \tag{21}$$

and,for the operators to be defined in (40) and (45) in Section 2[11],

$$ker\ A_s \ = \ ker\ a_s = \{|0\rangle\}$$

$$ker\ A_s^\dagger \ = \ ker\ a_s^\dagger = \{|s\rangle\} \tag{22}$$

The choice of A_s is not unique for a given a_s. The index relation (21),which holds for arbitrary s, is then regarded as a manifestation of the invariance of the index under a change of the deformation parameter s. The limit $s \to \infty$ of the relation (21)(and also (20))is however singular since the kernels in (22) are ill-defined for $s \to \infty$. We thus analyze the case of arbitrarily large but $finite$ s in the main part of this paper; this is presumably the case in which the phase operator of Pegg and Barnett may have some practical use. The limit $s \to \infty$ of (20) will be discussed in Section 5 in connection with quantum anomaly.

To assign a precise meaning to this statement of arbitrarily large s, we expand a given *physical* state $|p\rangle$ as

$$|p\rangle = \sum_{n=0}^{\infty} p_n |n\rangle \tag{23}$$

which is assumed to give a finite $\langle p|N^2|p\rangle$

$$\langle p|N^2|p\rangle = \sum_{n=0}^{\infty} n^2 |p_n|^2 = N_p^2 < \infty \tag{24}$$

in addition to the normalizability of a vector in a Hilbert space

$$\langle p|p\rangle = \sum_{n=0}^{\infty} |p_n|^2 < \infty \tag{25}$$

The number N_p in (24) characterizes a given physical system, and $s >> N_p$ specifies a sufficiently large but finite s.

The existence of the unitary phase factor $e^{i\phi}$ of Pegg and Barnett or its infinite dimensional extension $e^{i\Phi}$,which is associated with A_s, critically depends on the vanishing index in (20) and

(21), which is in turn specified by the state $|s\rangle$ in the kernels in (22). This fact suggests that the state $|s\rangle$, whatever large s may be, critically influences the algebraic relations satisfied by the operator $e^{i\phi}$ and, consequently, the absence or presence of minimum uncertainty states for the operator ϕ. In Sections 3 and 4, we shall in fact show that the state $|s\rangle$, which characterizes the index in (20) , causes a substantial deviation from minimum uncertainty for the phase operator of Pegg and Barnett in a characteristically quantum domain with small average photon numbers.

In Section 5, we discuss a close analogy between the present problem of the phase operator ,which is related to the non-trivial index in (7), and the chiral anomaly in gauge theory which is related to the Atiyah-Singer index theorem.

2 Quantum Phase Operators and Uncertainty Relations

We first summarize the definitions and basic properties of two representative "phase" operators, namely, the one due to Susskind and Glogower[4] and the other due to Pegg and Barnett[8]. The operator suggested by Susskind and Glogower is

$$e^{i\varphi} = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3| + ... \tag{26}$$

in terms of the eigenstates $|k\rangle$ of the number operator in (6) This phase operator is related to the operator a in (5) by $a = e^{i\varphi}N^{1/2}$. The analogues of cosine and sine functions are then defined by

$$C(\varphi) \;=\; \frac{1}{2}(e^{i\varphi} + (e^{i\varphi})^{\dagger})$$

$$S(\varphi) = \frac{1}{2i}(e^{i\varphi} - (e^{i\varphi})^\dagger) \tag{27}$$

Note that $e^{i\varphi}$ is a symbolic notation since $e^{i\varphi}$ is not unitary and φ is not defined as a hermitian operator, as is witnessed by

$$\begin{aligned} (e^{i\varphi})^\dagger e^{i\varphi} &= 1 - |0\rangle\langle 0| \\ e^{i\varphi}(e^{i\varphi})^\dagger &= 1 \end{aligned} \tag{28}$$

One can write the operator $e^{i\varphi}$ in terms of the operator a in (5) as

$$e^{i\varphi} = \frac{1}{\sqrt{N+1}}a \tag{29}$$

As long as one considers a representation with non-negative N in (29),one obtains the index relation[11]

$$\begin{aligned} & dim\ ker\ e^{i\varphi} - dim\ ker\ (e^{i\varphi})^\dagger \\ = \ & dim\ ker\ a - dim\ ker\ a^\dagger = 1 \end{aligned} \tag{30}$$

namely, the operator $e^{i\varphi}$ carries a unit index which can be confirmed by the explicit expression in (26). The index relation (30) is also written as[11]

$$dim\ ker\ (e^{i\varphi})^\dagger e^{i\varphi} - dim\ ker\ e^{i\varphi}(e^{i\varphi})^\dagger = 1 \tag{31}$$

This last form of index relation is directly related to eq.(28),which is in turn responsible for an anomalous commutator

$$[C(\varphi), S(\varphi)] = \frac{1}{2i}|0\rangle\langle 0| \tag{32}$$

and an anomalous identity

$$C(\varphi)^2 + S(\varphi)^2 = 1 - \frac{1}{2}|0\rangle\langle 0| \tag{33}$$

satisfied by the hermitian operators in (27). One thus sees a relation between the non-trivial index of a in (7) and the anomalous behaviour of $C(\varphi)$ and $S(\varphi)$. The modified trigonometric operators also satisfy the commutation relations with the number operator N,

$$\begin{aligned} [N, C(\varphi)] &= -iS(\varphi) \\ [N, S(\varphi)] &= iC(\varphi) \end{aligned} \tag{34}$$

On the other hand, the genuine hermitian phase operator ϕ of Pegg and Barnett[8] is defined in a truncated $s+1$ dimensional space by

$$e^{i\phi} = |0\rangle\langle 1| + |1\rangle\langle 2| + ... + |s-1\rangle\langle s| + e^{i(s+1)\phi_0}|s\rangle\langle 0| \tag{35}$$

where ϕ_0 is an arbitrary constant c-number. The operator $e^{i\phi}$ is in fact unitary in an $s+1$ dimension

$$e^{i\phi}(e^{i\phi})^\dagger = (e^{i\phi})^\dagger e^{i\phi} = 1 \tag{36}$$

One may then define cosine and sine operators by

$$\begin{aligned} \cos\phi &= \frac{1}{2}(e^{i\phi} + e^{-i\phi}) \\ \sin\phi &= \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \end{aligned} \tag{37}$$

with $e^{-i\phi} = (e^{i\phi})^\dagger$. These operators together with the number operator satisfy the commutation relations

$$\begin{aligned} [N, \cos\phi] &= -i\sin\phi \\ &+ \frac{(s+1)}{2}[e^{i(s+1)\phi_0}|s\rangle\langle 0| - e^{-i(s+1)\phi_0}|0\rangle\langle s|] \\ [N, \sin\phi] &= i\cos\phi \\ &- i\frac{(s+1)}{2}[e^{i(s+1)\phi_0}|s\rangle\langle 0| + e^{-i(s+1)\phi_0}|0\rangle\langle s|] \\ [\cos\phi, \sin\phi] &= 0 \end{aligned} \tag{38}$$

and

$$\cos^2\phi + \sin^2\phi = 1 \tag{39}$$

The $s+1$ dimensional truncated operator defined by

$$a_s = e^{i\phi}(N)^{1/2} = |0\rangle\langle 1| + |1\rangle\langle 2|\sqrt{2} + ... + |s-1\rangle\langle s|\sqrt{s} \tag{40}$$

and its hermitian conjugate $a_s^\dagger$ satisfy the algebra

$$[a_s, a_s^\dagger] = 1 - (s+1)|s\rangle\langle s| \tag{41}$$

which suggests $a_s a_s^\dagger |s\rangle = 0$, and thus leads to the index condition(20)

$$dim\ ker\ a_s^\dagger a_s - dim\ ker\ a_s a_s^\dagger = 0 \tag{42}$$

as is required by a general analysis (13). The operator $e^{i\phi}$ cannot be expressed in a form analogous to (29),but the unitary operator inevitably carries a trivial index

$$\begin{aligned} & dim\ ker\ e^{i\phi} - dim\ ker\ (e^{i\phi})^\dagger \\ = \ & dim\ ker\ (e^{i\phi})^\dagger e^{i\phi} - dim\ ker\ e^{i\phi}(e^{i\phi})^\dagger = 0 \end{aligned} \tag{43}$$

The unitary operator simply re-labels the names of basis vectors with the number of basis vectors kept fixed; namely, $dim\ ker\ e^{i\phi} = dim\ ker\ (e^{i\phi})^\dagger = 0$. One can confirm that the state $|s\rangle$ plays a crucial role in specifying the indices of a_s in (42) and $e^{i\phi}$ in (35); the trivial index of $e^{i\phi}$ is related to the unitary property of $e^{i\phi}$,which in turn leads to the normal algebraic relations between $cos\phi$ and $sin\phi$ in (38) and (39).

From a view point of index theory, it is convenient to define an infinite dimensional operator $e^{i\Phi}$ which has the same characteristics as $e^{i\phi}$ in (35). One possible choice may be

$$e^{i\Phi} \ = \ e^{i\phi}$$

$$\begin{aligned}
&+ \quad |s+1\rangle\langle s+2| + ... + |2s\rangle\langle 2s+1| \\
&+ \quad e^{i\phi_1}|2s+1\rangle\langle s+1| \\
&+ \quad |2s+2\rangle\langle 2s+3| + ... + |3s+1\rangle\langle 3s+2| \\
&+ \quad e^{i\phi_2}|3s+2\rangle\langle 2s+2| + ...
\end{aligned} \tag{44}$$

where $e^{i\phi}$ is given in (35) and $\phi_1, \phi_2, ...$ are real constants. This operator is unitary $e^{i\Phi}(e^{i\Phi})^\dagger = (e^{i\Phi})^\dagger e^{i\Phi} = 1$, and $\langle k|e^{i\Phi}|k\rangle = 0$ for any k.The operator $e^{i\Phi}$ gives rise to

$$\begin{aligned}
A_s &= e^{i\Phi}(N)^{1/2} \\
&= a_s \\
&+ |s+1\rangle\langle s+2|\sqrt{s+2} + ... + |2s\rangle\langle 2s+1|\sqrt{2s+1} \\
&+ e^{i\phi_1}|2s+1\rangle\langle s+1|\sqrt{s+1} \\
&+ |2s+2\rangle\langle 2s+3|\sqrt{2s+3} + ...
\end{aligned} \tag{45}$$

where a_s stands for the operator in (40).

One can then confirm the index relation

$$Tr(e^{-A_s^\dagger A_s/M^2}) - Tr(e^{-A_s A_s^\dagger/M^2}) = 0 \tag{46}$$

to be consistent with (13). Apparently, A_s is not unitary equivalent to a in (5) for arbitrarily large s; the limit $s \to \infty$ is however a singular point of (46) since $ker\ A_s^\dagger = ker\ a_s^\dagger = \{|s\rangle\}$ is ill-defined in this limit. In our explicit analyses of uncertainty relations in the next section, $e^{i\phi}$ and $e^{i\Phi}$ give rise to the same physical consequences for sufficiently large but finite s for any physical state which satisfies (24).

We next recapitulate a derivation of the Heisenberg uncertainty relation for the commutator

$$[A, B] = iC \tag{47}$$

where A, B and C are hermitian operators. The expectation values of these operators, which are real numbers, are given by

$$\begin{aligned}\langle A\rangle &= \langle p|A|p\rangle \\ \langle B\rangle &= \langle p|B|p\rangle \\ \langle C\rangle &= \langle p|C|p\rangle\end{aligned} \tag{48}$$

for a suitable state $|p\rangle$, which is assumed to give well-defined expectation values in (48). We then define the operators

$$\begin{aligned}\hat{A} &= A - \langle A\rangle \\ \hat{B} &= B - \langle B\rangle\end{aligned} \tag{49}$$

which satisfy the same algebra as (47)

$$[\hat{A}, \hat{B}] = iC \tag{50}$$

We consider a function $f(t)$ defined by

$$f(t) = \sum_n |\langle n|\hat{A}|p\rangle - it\langle n|\hat{B}|p\rangle|^2 \tag{51}$$

where t is a real parameter and the sum over n runs over all the eigenstates of the number operator. The function $f(t)$ is rewritten as

$$\begin{aligned}f(t) &= \sum_n [\langle p|\hat{A}|n\rangle\langle n|\hat{A}|p\rangle + t^2\langle p|\hat{B}|n\rangle\langle n|\hat{B}|p\rangle \\ &\quad -it(\langle p|\hat{A}|n\rangle\langle n|\hat{B}|p\rangle - \langle p|\hat{B}|n\rangle\langle n|\hat{A}|p\rangle)] \\ &= t^2\langle p|\hat{B}^2|p\rangle + t\langle p|C|p\rangle + \langle p|\hat{A}^2|p\rangle\end{aligned} \tag{52}$$

where we used the hermiticity of $\hat{A}$ and $\hat{B}$, and also the algebra (50). By definition, $f(t)$ is positive semi-definite

$$f(t) \geq 0 \tag{53}$$

as a quadratic function of real variable t. Consequently, the discriminant of $f(t)$ is non-negative

$$D = \langle p|\hat{A}^2|p\rangle\langle p|\hat{B}^2|p\rangle - \frac{1}{4}(\langle p|C|p\rangle)^2 \geq 0 \tag{54}$$

which gives rise to the uncertainty relation

$$\Delta A\Delta B \geq \frac{1}{2}|\langle p|C|p\rangle| \tag{55}$$

if one defines ΔA and ΔB by

$$\begin{aligned} (\Delta A)^2 &\equiv \langle p|\hat{A}^2|p\rangle = \langle p|A^2|p\rangle - (\langle p|A|p\rangle)^2 \\ (\Delta B)^2 &\equiv \langle p|\hat{B}^2|p\rangle = \langle p|B^2|p\rangle - (\langle p|B|p\rangle)^2 \end{aligned} \tag{56}$$

The necessary and sufficient condition for the equality in (55) (i.e., the minimum uncertainty state)[13] is

$$\langle n|\hat{A}|p\rangle - it\langle n|\hat{B}|p\rangle = 0 \tag{57}$$

for <u>all</u> n with a fixed real t given by

$$t = -\frac{\langle p|C|p\rangle}{2\langle p|\hat{B}^2|p\rangle} = -\frac{2\langle p|\hat{A}^2|p\rangle}{\langle p|C|p\rangle} \tag{58}$$

To analyze the uncertainty relation involving the number operator N ,the state $|p\rangle$ in the Hilbert space is required to satisfy the condition (24), namely, $\langle p|N^2|p\rangle = \sum_n n^2|p_n|^2 < \infty$. We call such states $|p\rangle$ as "physical states" hereafter. The condition (24) in particular suggests

$$\lim_{n\to\infty} n^3|p_n|^2 = 0 \tag{59}$$

3 Minimum Uncertainty States and Index Condition

We explain why the presence or absence of the minimum uncertainty state can be a good characteristic of two different phase operators with different indices. For this purpose, we start with an analysis of the construction of matrix elements of various operators. In particular, we briefly explain how infinite dimensional operators are defined starting from finite domensional ones. By this way, one can clearly understand a special role played by the state $|s\rangle$ in the hermitian phase operator $e^{i\phi}$.

We thus analyze a set of matrix elements defined for sufficiently large but finite s

$$\{\langle n|O|p\rangle\ |n \in \Sigma_s\} \tag{60}$$

where the operator O generically stands for one of $(s+1)$ dimensional operators, a_s, $a_s^\dagger$ in (40) and the phase variables $\cos\phi$ and $\sin\phi$ defined in (37). The state $|p\rangle$ is any physical state satisfying (24): To be precise one may have to cut-off the states in (23) at p_s, but this does not influence our analysis below. The matrix elements of operators such as $a^\dagger a$ and $aa^\dagger$ are then constructed from the matrix elements in (60). (An analysis of more general cases than in (60) is possible, but the analysis of (60) is sufficient for the discussion of uncertainty relations in the present paper). Σ_s is a set of non-negative integers smaller than $s+1$

$$\Sigma_s = \{1, 2, ..., s\} \tag{61}$$

with s standing for a cut-off parameter.

The conventional oscillator variables and the associated phase operator , which lead to the index relation (7), are specified by

the set of matrix elements

$$\{\langle n|O|p\rangle \,|n \in \Sigma_s'\} \tag{62}$$

in the limit $s \to \infty$, where Σ_s' is <u>any subset</u> of Σ_s ($\Sigma_s' \neq \Sigma_s$), which covers all the non-negative integers in the limit $s \to \infty$. For example, $\Sigma_s' = \Sigma_{s-1}$ or $\Sigma_{[s/2]}$ with $[s/2]$ the largest integer not exceeding $s/2$, etc. This specification of the operators presumes a <u>uniform</u> convergence of the set of matrix elements (62) with respect to the choice of Σ_s' for $s \to \infty$. In other words, one abstracts only those properties which are independent of the precise value of the cut-off parameter s when $s \to \infty$. In the present case, one can confirm that the set of matrix elements thus defined for $\cos\phi$, for example,in fact reproduces the set of matrix elements of the infinite dimensional operator $C(\varphi)$ in (27) . The hermiticity of the phase operator ϕ is lost in this limiting procedure and ϕ is converted to φ. (A related analysis from a view point of quantum anomaly is found in Section 5).

On the other hand,the truncated space which leads to the representation of a and $a^\dagger$ with an index 0 to ensure the presence of a hermitian phase operator is specified by (60) ;one uses a <u>very specific</u> Σ_s for $s \to large$ and the uniformity of the convergence of the set of matrix elements is not satisfied. The characteristic property of the operators specified by Σ_s, which includes $|s\rangle$, arises from the fact that for arbitrarily large s

$$\langle s|\cos\phi|p\rangle \neq 0 \tag{63}$$

in general for $\cos\phi$, and similarly for $\sin\phi$,in (37). Since one cannot discriminate $|s-1\rangle$ from $|s\rangle$ in the limit $s \to \infty$, the limit $s \to \infty$ is not well-defined in the present case. See papers in refs.[9] and [10] for the discussions of this problem . In the

present paper we simply keep s arbitrarily large but finite for the operators specified by Σ_s and analyze their physical implications.

The state $|s\rangle$ is responsible for the different indices of $e^{i\varphi}$ and $e^{i\phi}$, as was explained in Section 2, and also for the difference between the algebraic relations satisfied by φ in (32) and (33) and the algebraic relations satisfied by ϕ in (38) and (39). The absence or presence of minimum uncertainty states is related to those algebraic properties, and thus it becomes a good characteristic of operators carrying different indices.

We next examine the minimum uncertainty state by choosing A in (47) as the number operator

$$\hat{A} = \hat{N} = N - \langle p|N|p\rangle \tag{64}$$

and B as one of the phase operators;to be specific, we choose $C(\varphi)$ or $\cos\phi$ in (27) and (37), respectively. The condition for the minimum uncertainty state in (57) then becomes

$$\langle n|\hat{N}|p\rangle - it\langle n|\hat{C(\varphi)}|p\rangle = 0 \tag{65}$$

for all n, or

$$\langle n|\hat{N}|p\rangle - it'\langle n|\hat{\cos}\phi|p\rangle = 0 \tag{66}$$

for all $n \leq s$. Here the real parameters t and t' are generally different. In (66) we first fix s and impose the relation for <u>all</u> $n = 0 \sim s$, and later s is set to arbitrarily large compared to the typical number N_p associated with the physical state $|p\rangle$. (If one uses an infinite dimensional $\cos\Phi$, which is defined in terms of $e^{i\Phi}$ (44), in (66) instead of $\cos\phi$, one can treat (65) and (66) in a unified manner; $\cos\phi$ and $\cos\Phi$ give rise to the same physical results for a physical state $|p\rangle$ for sufficiently large s).

One expects that it is easier to satisfy the condition (65) than the condition (66), since the matrix elements $\langle n|\hat{N}|p\rangle$ and $\langle n|C(\hat{\varphi})|p\rangle$ are defined such that a uniform convergence for $n \to \infty$ is ensured. On the other hand, the matrix element $\langle n|\hat{\cos}\phi)|p\rangle$ of Pegg and Barnett critically depends on the specific number $n = s$, whatever large s may be, while s does not play a special role in $\langle n|\hat{N}|p\rangle$.

We now formulate the above qualitative consideration in a more explicit quantitative way. By recalling the definitions in (27) and (37), one can confirm that the expectation values of two different phase operators for a physical state are identical for sufficiently large s,

$$\begin{aligned}\langle p|C(\varphi)|p\rangle &= \langle p|\cos\phi|p\rangle \\ \langle p|S(\varphi)|p\rangle &= \langle p|\sin\phi|p\rangle\end{aligned} \tag{67}$$

The variation of the number operator N is of course common to both cases for sufficiently large s, and it is given by

$$(\Delta N)^2 = \langle p|N^2|p\rangle - (\langle p|N|p\rangle)^2 \tag{68}$$

However, the expectation values of the square of phase variables are generally different

$$\begin{aligned}\langle p|C(\varphi)^2|p\rangle &= \sum_{n=0}^{n_p}\langle p|C(\varphi)|n\rangle\langle n|C(\varphi)|p\rangle \\ \langle p|\cos^2\phi|p\rangle &= \sum_{n=0}^{n_p}\langle p|cos\phi|n\rangle\langle n|cos\phi|p\rangle \\ &\quad +\langle p|cos\phi|s\rangle\langle s|cos\phi|p\rangle \\ &= \sum_{n=0}^{n_p}\langle p|C(\varphi)|n\rangle\langle n|C(\varphi)|p\rangle\end{aligned}$$

$$
\begin{aligned}
& +|\langle p|\cos\phi|s\rangle|^2 \\
\geq\ & \langle p|C(\varphi)^2|p\rangle
\end{aligned}
\tag{69}
$$

where n_p stands for the maximum occupation number which contributes to the matrix element of $C(\varphi)$ sizably for a given physical state $|p\rangle$: One may choose $n_p >> N_p$ in (24). We also used the fact that the operator $\cos\phi$ is hermitian and that $\langle n|\cos\phi|p\rangle = \langle n|C(\varphi)|p\rangle$ for $n \leq n_p$ and sufficiently large s . Note that $\langle s|\cos\phi|p\rangle$ is not zero in general even for arbitrarily large s due to the definition in (35).

We thus conclude from (67) and (69) that

$$
\begin{aligned}
(\Delta\cos\phi)^2 &= (\Delta C(\varphi))^2 + |\langle p|\cos\phi|s\rangle|^2 \\
&\geq (\Delta C(\varphi))^2
\end{aligned}
\tag{70}
$$

and

$$
\begin{aligned}
\Delta N \Delta\cos\phi &\geq \Delta N \Delta C(\varphi) \\
&\geq \frac{1}{2}|\langle p|S(\varphi)|p\rangle| \\
&= \frac{1}{2}|\langle p|\sin\phi|p\rangle|
\end{aligned}
\tag{71}
$$

where we used the fact that

$$
\langle p|(\frac{s+1}{2})\{e^{i(s+1)\phi_0}|s\rangle\langle 0| - e^{-i(s+1)\phi_0}|0\rangle\langle s|\}|p\rangle = 0 \tag{72}
$$

in (38) for sufficiently large s by noting (59).

Similarly, one can derive from (32), (34) and (38)

$$
\begin{aligned}
\Delta N \Delta\sin\phi &\geq \Delta N \Delta S(\varphi) \\
&\geq \frac{1}{2}|\langle p|C(\varphi)|p\rangle|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}|\langle p|\cos\phi|p\rangle| &(73)\\
\Delta\cos\phi\Delta\sin\phi &\geq \Delta C(\varphi)\Delta S(\varphi) \\
&\geq \frac{1}{4}|\langle p|0\rangle\langle 0|p\rangle| \\
&\geq 0 &(74)
\end{aligned}
$$

From these relations we learn that the uncertainty relations are always better satisfied for the phase varibles $C(\varphi)$ and $S(\varphi)$. If the state $|p\rangle$ is a minimum uncertainty state for the variables $(\cos\phi, N)$ or $(\sin\phi, N)$,the same state automatically becomes a minimum uncertainty state for the variables $(C(\varphi), N)$ or $(S(\varphi), N)$,respectively. But the other way around is not true in general. Also, Eq.(74) shows that uncertainty in $\Delta\cos\phi\Delta\sin\phi$ is always greater than or equal to uncertainty in $\Delta C(\varphi)\Delta S(\varphi)$ for <u>any</u> given physical state $|p\rangle$ despite of $[\cos\phi, \sin\phi] = 0$ in (38). As far as the measurements of physical matrix elements are concerned, the non-commuting property of $C(\varphi)$ and $S(\varphi)$ provides a constraint less stringent than the commuting property of $\cos\phi$ and $\sin\phi$ which comes in pairs with $\cos^2\phi + \sin^2\phi = 1$. As for the minimum uncertainty states, it has been shown explicitly by Jackiw[5] that a minimum uncertainty state exists(under certain conditions) for the pair of variables $(N, C(\varphi))$ or $(N, S(\varphi))$, but there is no normalizable state which satisfies the minimum uncertainty relation (in the strict sense) for the pair $(C(\varphi), S(\varphi))$ in (74)[6].

The uncertainty relations for the hermitian variable ϕ of Pegg and Barnett substantially deviate from the minimum uncertainty when the physical state $|p\rangle$ has a substantial overlap with the vacuum state $|0\rangle$, since in this case $\langle p|\cos\phi|s\rangle$ or $\langle p|\sin\phi|s\rangle$ becomes appreciable in (70) or in a corresponding relation. In

a characteristically quantum domain with small particle numbers(or we use the term ”photon numbers” hereafter), we generally have no minimum uncertainty state for the phase operator of Pegg and Barnett except for the obvious case $\Delta N = 0$.

4 Minimum Uncertainty States in Characteristically Quantum Domain

To explicitly illustrate the absence of minimum uncertainty states for the phase variable ϕ in the characteristically quantum domain , we consider the case where the average photon number is almost zero and the probability of having one photon is very small with negligible probability for more than one photons. The most general state for this situation is given by

$$|\alpha\rangle \simeq \frac{1}{\sqrt{1+|\alpha|^2}}[|0\rangle + e^{i\theta}|\alpha||1\rangle] \tag{75}$$

with $|\alpha| \ll 1$, which incidentally corresponds to a small $|\alpha|$ limit of the coherent state

$$|\alpha\rangle \equiv e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{e^{in\theta}|\alpha|^n}{\sqrt{n!}}|n\rangle \tag{76}$$

For the state $|\alpha\rangle$ in (75), one has

$$\begin{aligned}
\langle\alpha|N|\alpha\rangle &= \frac{|\alpha|^2}{1+|\alpha|^2} \\
\langle\alpha|N^2|\alpha\rangle &= \frac{|\alpha|^2}{1+|\alpha|^2} \\
\langle\alpha|C(\varphi)|\alpha\rangle &= \langle\alpha|\cos\phi|\alpha\rangle = \frac{|\alpha|}{1+|\alpha|^2}\cos\theta
\end{aligned}$$

$$
\begin{aligned}
&\simeq |\alpha|\cos\theta \\
\langle\alpha|S(\varphi)|\alpha\rangle &= \langle\alpha|\sin\phi|\alpha\rangle = \frac{|\alpha|}{1+|\alpha|^2}\sin\theta \\
&\simeq |\alpha|\sin\theta \\
\langle\alpha|C(\varphi)^2|\alpha\rangle &\simeq \frac{1}{4} \\
\langle\alpha|\cos^2\phi|\alpha\rangle &= \frac{1}{2} \qquad (77)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta N &= \sqrt{\langle\alpha|N^2|\alpha\rangle - \langle\alpha|N|\alpha\rangle^2} \\
&\simeq |\alpha| \\
\Delta C(\varphi) &= \sqrt{\frac{1}{4} - \langle\alpha|C(\varphi)|\alpha\rangle^2} \\
&\simeq \frac{1}{2} \\
\Delta\cos\phi &= \sqrt{\frac{1}{2} - \langle\alpha|\cos\phi|\alpha\rangle^2} \\
&\simeq \sqrt{\frac{1}{2}} \qquad (78)
\end{aligned}
$$

One therefore obtains

$$
\begin{aligned}
\Delta N\Delta C(\phi) &\simeq \frac{1}{2}|\alpha| \\
\Delta N\Delta\cos\phi &\simeq \sqrt{\frac{1}{2}}|\alpha| \qquad (79)
\end{aligned}
$$

and uncertainty relations by noting (72)

$$
\Delta N\Delta\cos\phi = \sqrt{\frac{1}{2}}|\alpha| > \Delta N\Delta C(\varphi) = \frac{1}{2}|\alpha| \;\geq\; \frac{1}{2}|\alpha||\sin\theta| \qquad (80)
$$

with $|\alpha||\sin\theta| = |\langle\alpha|S(\varphi)|\alpha\rangle| = |\langle\alpha|\sin\phi|\alpha\rangle|$. The minimum uncertainty is achieved for $|\sin\theta| = 1$ for the set of variables $(N, C(\varphi), S(\varphi))$, but no minimum uncertainty state for the set of variables $(N, \cos\phi, \sin\phi)$ of Pegg and Barnett.

Similarly one obtains

$$\begin{aligned} \Delta N \Delta \sin\phi = \sqrt{\frac{1}{2}}|\alpha| &> \Delta N \Delta S(\varphi) = \frac{1}{2}|\alpha| \geq \frac{1}{2}|\alpha||\cos\theta| \\ \Delta\cos\phi\Delta\sin\phi = \frac{1}{2} &> \Delta C(\varphi)\Delta S(\varphi) = \frac{1}{4} = \frac{1}{4}|\langle\alpha|0\rangle|^2 \end{aligned} \tag{81}$$

Note that the uncertainty product $\Delta\cos\phi\Delta\sin\phi = 1/2$ is essentially a consequence of $\cos^2\phi + \sin^2\phi = 1$. The measurement of the uncertainty product in the second equation of (81) may provide a direct experimental test of the choice of a physical phase operator.

Physically, a marked deviation from minimum uncertainty for the variable ϕ may be understood as follows: To maintain the hermiticity of ϕ and make ϕ rotate over full 2π angle in a characteristically quantum domain, the transition from $|0\rangle$ to $|s\rangle$ is required as is seen from the last term in (35). This is not possible as a real physical process for large s. On the other hand, all the states up to $|s\rangle$ contribute to the intermediate states of algebraic relations such as (38) and (39) without any suppression factor. These two properties combined cause a severe discrepancy between the physical matrix elements and the formal algebraic relations, as is seen in (80) and (81).In passing, we note that the transition from $|s+1\rangle$ to $|2s+1\rangle$, for example, in $e^{i\Phi}$ (44) is negligible for a physical state which satisfies (59), and thus the two hermitian phase operators $e^{i\phi}$ and $e^{i\Phi}$ give the

same physical results in this section.

This absence of the minimum uncertainty state for the phase variable of Pegg and Barnett is shown more generally on the basis of relations (70) and (71). A necessary condition for the minimum uncertainty for the variable of Pegg and Barnett is that the physical states $|p\rangle$ satisfy

$$\langle s|\cos\phi|p\rangle = 0 \textit{ or } \langle s|\sin\phi|p\rangle = 0 \tag{82}$$

namely, $|p\rangle$ do not contain the zero photon state $|0\rangle$, or the states $|p\rangle$ spread over many eigenstates of the number operator such that the term $|\langle s|\cos\phi|p\rangle|^2$ is negligible compared with $\sum_{n=0}^{n_p}|\langle n|\cos\phi|p\rangle|^2$ in (69). This latter possibility is realized, for example, by the coherent state with large $|\alpha|$.[8]

In passing, the algebraic consistency is improved for the variable of Pegg and Barnett if one sets the cut-off parameter s in the region of average photon number. If one sets $s = 1$, for example, one obtains

$$\begin{aligned}\langle\alpha|\cos\phi|\alpha\rangle &= 2|\alpha|\cos\phi_0\cos(\theta-\phi_0)\\ \langle\alpha|\cos^2\phi|\alpha\rangle &\simeq \cos^2\phi_0\end{aligned} \tag{83}$$

and, consequently, from (38) with $s = 1$

$$\begin{aligned}\Delta N\Delta\cos\phi &\simeq |\alpha||\cos\phi_0|\\ &\geq \frac{1}{2}|\langle\alpha|\{\sin\phi + ie^{2i\phi_0}|1\rangle\langle 0| - ie^{-2i\phi_0}|0\rangle\langle 1|\}|\alpha\rangle|\\ &= |\alpha||\cos\phi_0\sin(\theta-\phi_0)|\end{aligned} \tag{84}$$

The minimum uncertainty is achieved if

$$|\sin(\theta-\phi_0)| = 1 \tag{85}$$

By recalling the $(s+1)$ orthonormal eigenstates of the phase variable ϕ[8]

$$|\phi_m\rangle = (s+1)^{-1/2}\sum_{n=0}^{s} e^{in\phi_m}|n\rangle \tag{86}$$

with $\phi_m = \phi_0 + 2\pi m/(s+1)$, $m = 0 \sim s$, one obtains for $s = 1$

$$\begin{aligned} P(\phi_0) = |\langle\phi_0|\alpha\rangle|^2 &= \frac{1}{2}[1 + 2|\alpha|\cos(\theta - \phi_0)] \\ P(\phi_1) = |\langle\phi_1|\alpha\rangle|^2 &= \frac{1}{2}[1 - 2|\alpha|\cos(\theta - \phi_0)] \end{aligned} \tag{87}$$

For the choice of $\theta - \phi_0$ in (85), one has a uniform phase distribution

$$P(\phi_0) = P(\phi_1) = 1/2 \tag{88}$$

This exercise shows that the algebraic consistency is improved for $s \sim 1+|\alpha|+|\alpha|^2$, but one is apparently dealing with a theory different from the original one in (5).

5 Analogy with Chiral Anomaly

The absence of the minimum uncertainty state for the operators of Pegg and Barnett in a characteristically quantum domain arises from their very definition and the index mismatch. This fact may not prohibit the use of the phase variable of Pegg and Barnett as an interpolating variable in practical analyses for finite s, but it at least shows that we cannot attach much physical significance to the deviation from minimum uncertainty in a characteristically quantum domain with small photon numbers. Our consideration shows that the notion of index or an index

theorem provides a powerful machinery in the analysis of the representation of linear operators.

We here comment on an interesting analogy between the present problem and the chiral anomaly[14][15] in gauge theory which is related to the Atiyah-Singer index theorem.

In gauge theory one deals with a (Euclidean) Dirac operator defined by

$$\not{D} = \sum_{\mu=1}^{4} \gamma^{\mu}(\frac{\partial}{\partial x^{\mu}} - igA_{\mu}^{a}T^{a}) \tag{89}$$

where γ^{μ} is the 4×4 anti-hermitian Dirac matrix with $\gamma_5 \equiv \gamma^4\gamma^1\gamma^2\gamma^3$, $A_{\mu}^{a}(x)$ is the non-Abelian Yang-Mills field ,g is the coupling constant and T^a is the hermitian generator of the gauge group. The non-zero index relation for $\not{D}_R = \not{D}(\frac{1+\gamma_5}{2})$ [16]

$$dim\ ker\ \not{D}_R - dim\ ker\ \not{D}_R^{\dagger} = \nu \neq 0 \tag{90}$$

with ν the Pontryagin index,which is expressed in terms of the gauge field $A_{\mu}^{a}(x)$, is used as an argument for the presence of the chiral anomaly: The Hilbert space for a single fermion inside the background gauge field with $\nu \neq 0$ cannot be unitary equivalent to the Hilbert space of a free fermion with $\nu = 0$. Here the left-hand side of (90) is specified by the difference in the number of eigenstates of γ_5

$$\gamma_5\varphi_n(x) = \pm\varphi_n(x) \tag{91}$$

for $\lambda_n = 0$ in the eigenvalue problem in 4-dimensional space-time

$$\not{D}\varphi_n(x) = \lambda_n\varphi_n(x),\ \int \varphi_n^{\dagger}(x)\varphi_m(x)\, d^4x = \delta_{n,m} \tag{92}$$

The interaction picture assumes the unitary equivalence of these two Hilbert spaces with $\nu \neq 0$ and $\nu = 0$, and thus interaction picture perturbation theory inevitably encounters a surprise (i.e.,anomaly).

In the actual analysis of the chiral anomaly, it is known[15][17] that a careful treatment of the ultraviolet cut-off, which is analogous to the parameter s in the present phase operator, is required to recognize the consequence of the index relation (90): The decoupling or the failure of decoupling of the ultraviolet cut-off from physical quantities needs to be analyzed with great care. The connection between the non-zero index and the chiral anomaly appears in a particularly transparent way in the Euclidean path integral formulation of anomalies, which is based on the analysis of single fermion states in a background gauge field [17]. As for physics aspects, the chiral anomaly, which is related to high energy behavior in the interaction picture, critically influences the low energy phenomena such as the radiative decay of the neutral pion $\pi^0 \to \gamma\gamma$ [15].

If one uses an analogy between the phase operator and the chiral anomaly, the index relation (7) corresponds to the presence of a quantum anomaly and the relation (13) or (20) to the normal situation naively expected by a classical consideration. The anomaly specified by the index relation (7) is clearly recognized only when one carefully analyzes the dependence of the matrix elements of various operators on the cut-off parameter s: For example,a sequence of the sets of matrix elements is not uniformly convergent in the formulation of Pegg and Barnett, as is seen in (60).

The normal situation (i.e., hermitian phase operator) realized by a finite dimensional formulation of Pegg and Barnett in (20) may then be regarded as corresponding to the case of chiral anomaly where the mass of the Pauli-Villars regulator is kept finite; the finite regulator mass generally avoids anomalous behavior but leads to a different theory. Also the effect of the

regulator does not quite decouple even in the limit of arbitrarily large regulator mass (or s), which is the origin of the discrepancy between the algebraic relations (32) and (38).

To be more explicit, one may rewrite the relation (20) as

$$\begin{aligned} & Tr_{(s+1)}(e^{-a_s^\dagger a_s/M^2}) - Tr_{(s)}(e^{-a_s a_s^\dagger/M^2}) \\ = \; & Tr_{(s+1)}(e^{-a_s a_s^\dagger/M^2}) - Tr_{(s)}(e^{-a_s a_s^\dagger/M^2}) = 1 \end{aligned} \tag{93}$$

where $Tr_{(s)}$ stands for the trace over the (first) s-dimensional subspace of the $(s+1)$-dimensional space; the right-hand side of (93) is the contribution of the state $|s\rangle$. This relation (93) holds for any positive M^2 and s, and it can be confirmed that each term in the left-hand side has a well-defined limit for $s \to \infty$, and one recovers (9) in this limit

$$Tr(e^{-a^\dagger a/M^2}) - Tr(e^{-aa^\dagger/M^2}) = 1 \tag{94}$$

This phenomenon is a precise analogue of the chiral anomaly; the effect of the cut-off, i.e.,the state $|s\rangle$, gives rise to the "anomaly" in the right-hand side of (94) in the limit $s \to \infty$, though one obtains a normal relation (20) for a finite s.

The physical implications of the phase operator anomaly appear most prominently in the "low energy" processes with small average photon numbers, such as in (80) and (81); apparently "anomalous" behavior exhibited by the phase operator which carries a unit index is in fact more consistent with quantum phenomena. In terms of operator language,the relation in (32) may be regarded as an anomalous commutator and the relation (33) as an anomalous identity: Namely we have

$$[C(\varphi), S(\varphi)] \;=\; \frac{1}{2i}|0\rangle\langle 0|$$

$$C(\varphi)^2 + S(\varphi)^2 \;=\; 1 - \frac{1}{2}|0\rangle\langle 0| \qquad (95)$$

both of which are characteristic properties of any quantum anomaly[15][18]. We also emphasized that these properties are directly related to the non-vanishing index in (31). One may thus regard the phase operator problem associated with the non-vanishing analytic index in (7) as a new class of quantum anomaly. From this view point , the anomalous behavior seen in (32) and (33) is an <u>inevitable</u> real quantum effect, not an artifact of our insufficient definition of phase operator. This may be tested by experiments by measuring the uncertainty product in (81) or its variants, since the phase operator of Susskind and Glogower, though quite an attractive choice, is one of those operators which carry a natural index, i.e.,a unit index.

We also mention that an analogue of anomaly in the present phase operator is based on an analysis of multi-photon states with a fixed momentum and polarization and thus it is characteristic to bosonic systems, whereas the conventional anomaly in field theory is primarily based on an analysis of one- particle states [17] which is applicable to both of fermions and bosons. The phase operator anomaly is a qualitatively new quantum anomaly.

It is interesting that the first paper on quantum field theory [1] already contained an analytic index and quantum anomaly.

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$dim\ ker\ a - dim\ ker\ a^{\dagger} = 1$
instead of (7), or
$dim\ ker\ a_s - dim\ ker\ a_s^{\dagger} = 0$
for the operator a_s in (20);$dim\ ker\ a$, for example, stands for the number of normalizable states which are annihilated by the operator a. In the body of the present paper we use the relations (7) and (20), since they have a more transparent physical interpretation as a difference in the number of zero eigenvalues of hermitian operators $a^{\dagger}a$ and $aa^{\dagger}$:The equivalence of these two specifications is seen by noting that

$au = 0$ implies $a^\dagger au = 0$. Conversely, $a^\dagger au = 0$ implies $(a^\dagger au, u) = (au, au) = 0$ and thus $au = 0$ if the inner product is positive definite.

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